Assam Academy of Mathematics

MATHLETICS, 2018

(Classes IX, X and X appeared)

Time : 3 Hours (10am to 1pm.) Marks : 100

1. $6 \times 3 = 18$

(i) Which of the following two numbers is greater

$\sqrt[3]{0.01}$ or $\sqrt[5]{0.001}$?

Solution : $\sqrt[3]{0.01} = (0.01)^{1/3}$

$$= \left[\left\{(0.01)^{5}\right\}^{1/5}\right]^{1/3}$$

$$= \left[\{0.0000000001\}^{1/3}\right]^{1/5}$$

$$< \left[\{0.000000001\}^{1/3}\right]^{1/5}$$

$$= \left[\left\{(0.001)^{3}\right\}^{1/3}\right]^{1/5}$$

$$= (0.001)^{1/5} = \sqrt[5]{0.001}$$

i.e. $\sqrt[5]{0.001}$ is greater than $\sqrt[3]{0.01}$

(ii) Solve the system of equations

$$\log_2 xy = 5, \quad \log_{\frac{1}{2}}\left(\frac{x}{y}\right) = 1$$

P. T. O.

Solution : Equations are

$\log_2 xy = 5$ and $\log_{\frac{1}{2}}\left(\frac{x}{y}\right) = 1$

ie. $xy = 2^5 = 32$ and $\frac{x}{y} = \frac{1}{2}$

ie. $xy = 32$ and $y = 2x$

$\Rightarrow x.\ 2x = 32 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$ and $y = \pm 8$

Required solutions are $x = 4, y = 8$ and $x = -4, y = -$

8

(iii) Prove that for all real x and y, the inequality

$x^2 + 2xy + 3y^2 + 2x + 6y + 3 \geq 0$ holds

Solution : $x^2 + 2xy + 3y^2 + 2x + 6y + 3$

$= x^2 + y^2 + 2xy + 2x + 2y + 1 + 2y^2 + 4y + 2$

$= (x + y + 1)^2 + 2(y^2 + 2y + 1)$

$= (x + y + 1)^2 + 2(y + 1)^2 \geq 0$, for all real x and y

(iv) Solve the inequation

$x+4 < -\frac{2}{x+1}$, where x is real

Solution :

Case -1.

Let $x + 1 > 0$

Then multiplying by $x + 1 > 0$

$(x + 1)(x + 4) < -2$

$\Rightarrow x^2 + 5x + 6 < 0$

$\Rightarrow (x + 2)(x + 3) < 0$

$\Rightarrow x + 2 > 0, x + 3 < 0$ or $x + 2 < 0, x + 3 > 0$

$\Rightarrow x > -2, x < -3$ or $x < -2, x > -3$

But $x > -2, x < -3$ is not possible

Hence feasible solution is $-3 < x < -2$

ie. $-3 + 1 < x + 1 < -2 + 1$

ie. $-2 < x + 1 < -1$

ie. $x + 1 < 0$ which contradicts the anumption that $x + 1 > 0$

Thus $x + 1 > 0$ is impossible.

Case -2.

Let $x + 1 < 0$

Then $(x + 1)(x + 4) > -2$

$\Rightarrow x^2 + 5x + 4 + 2 > 0$

$\Rightarrow x^2 + 5x + 6 > 0$

$\Rightarrow (x + 2)(x + 3) > 0$

$\Rightarrow x > -2, x > -3$ or $x < -2, x < -3$

$\Rightarrow x > -2$ or $x < -3$

But already we anumed $x < -1$

Hence required solution is given by –

$-2 < x < -1$ and $x < -3$

(v) Find all natural numbers n such that the fraction $\frac{3n+4}{5}$ is an integer

Solution :

$\frac{3n+4}{5}$ being an interger, $3n + 4$ must be an integral multiple of 5 , $3n + 4 = 5k$, k being interger.

$\Rightarrow 3n + 4 \equiv 0 \pmod 5$

$\Rightarrow 3n \equiv -4 \pmod 5$

$\equiv 1 \pmod 5$

$\equiv 6 \pmod 5$

$\Rightarrow n \equiv 2 \pmod 5$

$\Rightarrow n = 2 + 5k$, K any integer

n being natural number,

$n = 2, 7, 12, 17$ ....

(vi) How many four digit numbers with two middle digits 97 are divisible by 45?

Solution :

Let the four digit numbers be x 97y or x79y

These numbers being divisible by $45 = 5\times9$, are also

divisible by 5 and 9.

This requires the value of y to be either 0 or 5.

Suppose $y = 0$, then $x + 9 + 7 + 0 = x + 16$ is divisible by 9 which mean $x = 2$

In this case we have two numbers 2970 and 2790

Next let $y = 5$, then $x + 9 + 7 + 5$ is divisible by 9

ie. $x + 21$ is divigible by 9

ie. $x = 6$

In this case the required numbers are 6975 and 6795.

There are four four digit numbers with two middle digits 9 and 7 which are divisible by 45. These are 2970 , 2790, 6975 and 6795.

2. An evil king wrote three secret two digit numbers a, b and c. A handsome prince must name three numbers X, Y and Z after which the king will tell him the sum $aX + bY + cZ$. The prince then must name all three of the king's numbers. Otherwise, he will be executed. Help the prince get out of this dangerous situation. 6

Solution :

The prince must choose X, Y, Z in such a way that a, b and c remain visible in the sum $aX + bY + cZ$ as abc or acb or bac or bca or cab or cba. Confining ourselves to abc, $aX + bY + cZ$ may be considered as its expansion with respect to a certain base. a, b, c being two digit numbers, base for expansion of abc must be 100

Thus $abc = 100^2 a + 100b + 1c$

Hence $X = 100^2$, $Y = 100$ and $Z = 1$.

Thus by choosing X, Y, Z as above, the prince will be able to name all three of the kings numbers a, b and c.

3. Thirty students received marks 2, 3, 4, 5 at an examination. The sum of the marks is 93, three are more 3's than 5's and fewer 3's than 4's. besides, the number of 4's is divisible by 10 and the number of 5's is even. Determine the number of various marks received by the thirty students. 6

Solution :

Let the no of students receiving marks 2, 3, 4 and 5 be p, q, r and s respectively.

Thus $p + q + r + s = 30$ ---------- (i)

$2p + 3q + 4r + 5s = 93$ ---------- (ii)

$s < q < r$ ---------- (iii)

$p, q, r, s, \geq 1$

$r = 10t$ and $s = 2n$ for some $t, n \geq 1$ ---------- (iv)

(ii) $2(p + q + r + s) + (q + 2r + 3s) = 93$

$\Rightarrow q + 2r + 3s = 93 - 60 = 33$ ---------- (v)

$\Rightarrow q + 2r + 3s > q + 2q + 3s = 3q + 3s$

ie. $3(q + s) < q + 2r + 3s = 33$

$\Rightarrow q + s < 11$ ---------- (v)

also $q + 2r + 3s < q + 2r + 3q = 4q + 2r$

ie. $4q + 2r > q + 2r + 3s = 33$

$\Rightarrow 2q + r > \frac{33}{2} > 16$ ---------- (vi)

(v) $\Rightarrow 11 > q + s > s + s = 2s$

$\Rightarrow s < \frac{11}{2} \leq 5 \quad \Rightarrow 2n \leq 5$

$\Rightarrow n \leq 2$

$\Rightarrow n = 1$ or 2

$\Rightarrow s = 2$ or 4

(vi) $\Rightarrow 16 < 2q + r < 2r + r = 3r$

$\therefore 3r > 16$

$\Rightarrow r > \frac{16}{3} \geq 6$

$\Rightarrow r = 10, 20, 30 \ldots\ldots$

From (v) $q + 2r + 3s = 33$, it is clear that $r = 10$

Also $30 = p + q + r + s < p + r + 11 \quad \because q + s < 11$

$= p + 21$ by (v)

$\Rightarrow p > 30 - 21 = 9$

Agian from (v) $q + 20 + 3s = 33$

For $s = 2$, $q = 13 - 6 = 7$ and for $s = 4$

$\Rightarrow q = 13 - 12 = 1$

But $q > s = 4$

Hence $q \neq 1$ and therefore $s \neq 4$

ie. $s = 2$ and $q = 7$

Thus $p = 30 - q - r - s = 30 - 7 - 10 - 2 = 11$

No of students receiving 2 , 3, 4, 5 marks are respectively 11, 7, 10 and 2

4. Prove that for any real number x

$$x(x+1)(x+2)(x+3) \geq -1$$ 6

Solution :

We have

$$
\begin{aligned}
& x(x+1)(x+2)(x+3) \\
= \; & x(x+3)(x^2+3x+2) \\
= \; & (x^2+3x)(x^2+3x+2) \\
= \; & y(y+2) \qquad \text{putting } y = x^2+3x \\
= \; & (y^2+2y+1) - 1 \\
= \; & (y+2)^2 - 1
\end{aligned}
$$

≥ -1 for any real number x.

5. Given a triangle ABC find the locus of a point M in the plane of the triangle such that the areas of the triangle ABM and BMC are equal. 6

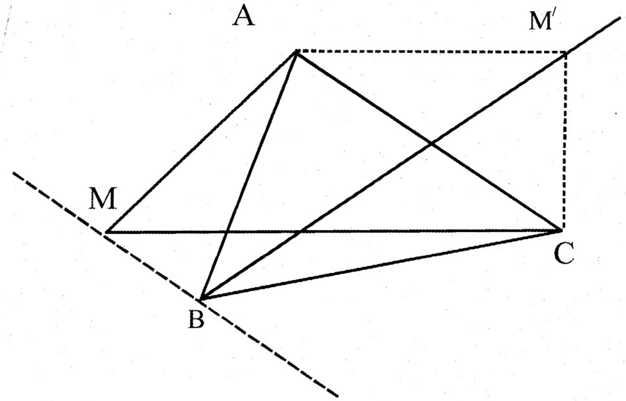
Solution:

Clearly B M will be common base for the triangle ABM and BMC. The triangle will have equal areas if the

perpendiculars from A and C to BM are equal.

This means that BM | | AC.

Hence Locus of M will be a straight line through B Parallel to AC.



Also let M′ be a point on the line through B joining the middle point of AC. Then also the perpendiculars from A and C to BM′ will be equal and therefore the areas of Δ ABM′ and Δ BM′C are again equal.

Hence the locus of M is a pair of straight lines passing throgh B, one parallel to AC and another joining the mid point of AC.

6. n straight lines are drawn in a plane such that no two of them are parallel and no three pass through a single point. Determine A_n, the number of parts into which the plane is divided. 6

Solution:

Let us compute A_n for smaller values of n like n = 1, 2, 3,, 4, 5...... and try to infer the general term An.

$A_1 = 2$

$A_2 = 4 = A_1 + 2$

$A_3 = 7 = 4 + 3 = A_2 + 3$

$A_4 = 11 = 7 + 4 = A_3 + 4$

$A_5 = 16 = 11 + 5 = A_4 + 5$

Thus

$A_1 = 2$

$A_2 = A_1 + 2$

$A_3 = A_2 + 3 = A_1 + 2 + 3$

$A_4 = A_3 + 4 = A_1 + 2 + 3 + 4$

$A_5 = A_4 + 5 = A_1 + 2 + 3 + 4 + 5$

In general

$$
\begin{aligned}
A_n &= A_1 + (2 + 3 + 4 + ... + n) \\
&= 2 + (2 + 3 + 4 + ... + n) \\
&= 1 + (1 + 2 + 3 + 4 + ... + n) \\
&= 1 + \frac{n(n+1)}{2} \\
&= \frac{1}{2}\left(n^2 + n + 2\right)
\end{aligned}
$$

7. How many integers strictly between 0 and 1,00000 have exactly one digit equal to 3. 8

Solution :

Considering leading zeros, each number strictly between 0 and 100000 can be considered as a string of five digits. For example 1 can be considered as 00001, 203 can be considered as 00203 etc. Since the numbers have exactly one digit equal to 3, so this 3 an occur in any one of the 5 position. For each such position of 3, there are exatly $9 \times 9 \times 9 \times 9 = 6561$ ways of placing the other digitd. Hence total number of such numbers is $5 \times 9 \times 9 \times 9 \times 9 = 32805$.

8. In how many ways can you climb up a staircase having 12 steps if you are allowed to take either one step or two steps at a time? 8

Solution:

Let s_n be the number of ways to climb n steps under the given condition. By observation, if there is only one step i.e. n = 1 then the number of ways to climb is 1.

ie. $s_1 = 1$

If n = 2, then there are two ways of climbing two 1 steps or one 2 step ie, $s_2 = 2$

If there are n steps, there are two options to choose in the beginning. If the first step is taken as a single step, then there are n – 1 more steps to climb. This can be done in S_{n-1} ways. If the first step is taken as 2 steps then there are n – 2 more steps to climb. This can be done in s_{n-2} ways. So the number of ways to climb n steps is

$s_n = s_{n-1} + s_{n-2}$

Now

$s_3 = s_1 + s_2 = 1 + 2 = 3$

$s_4 = s_2 + s_3 = 2 + 3 = 5$

$s_5 = s_3 + s_4 = 3 + 5 = 8$

$s_6 = s_4 + s_5 = 5 + 8 = 13$

$s_7 = s_5 + s_6 = 8 + 13 = 21$

$s_8 = s_6 + s_7 = 13 + 21 = 34$

$s_9 = s_7 + s_8 = 21 + 34 = 55$

$s_{10} = s_8 + s_9 = 34 + 55 = 89$

$s_{11} = s_9 + s_{10} = 55 + 89 = 144$

$s_{12} = s_{10} + s_{11} = 89 + 144 = 233$

9. How many sets of three numbers each can be formed from the numbers {1, 2, 3, ..., 20} if no two consecutive numbers are to be in the set? 8

Solution :

Total number of sets with three numbers from the given set is 20_{C_3}. There are exactly 19 two element subsets of the given set containing two consecutive numbers viz {1, 2}, {2, 3}, {3, 4} ......, {19, 20}. Thus two consecutive numbers can be Consecutive chosen in 19 way and for each such choice the third number can be chosen in 18 ways. So the number of three element subsets having at least two consecutive numbers is at most 19 ×18 but out of these there are some subsets

having three consecutive numbers and they have been counted twice. There are 18 such subsets viz {1, 2, 3}, {2, 3, 4}, {3, 4, 5}.... { 18, 19, 20}. Thus the number of three element subsets in which no two numbers are consecutive is

$$20_{C_3} - 19 \times 18 + 18 = 1140 - 342 + 18 = 816$$

10. Show that the number of three digit numbers of the form xzy such that $x<z$ and $z<y$ is 84. 8

Solution :

Clearly $2 \le z \le 8$

For z = n, there are exactly n – 1 choices viz 1 , 2, 3, ...n –1 for x and 9 – n choices viz n + 1, n + 2, ...9 for y.

So, total number of such numbers

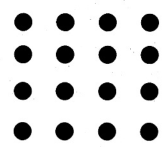
$$= \sum_{n=2}^{8}(n-1)(9-n)$$

$= 1 \times 7 + 2 \times 6 + 3 \times 5 + 4 \times 4 + 5 \times 3 + 6 \times 2 + 7 \times 1$

$= 7 + 12 + 15 + 16 + 15 + 12 + 7$

$= 84$

11. Consider an array of $2^n \times 2^n$ dots, with 2^n rows and 2^n columns, as shown in the figure below for n=2.



Reversing the process

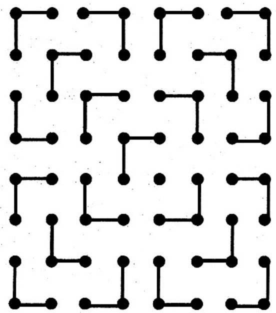


Fig 3

Putting one more L and reflecting the bottom right corner, we have

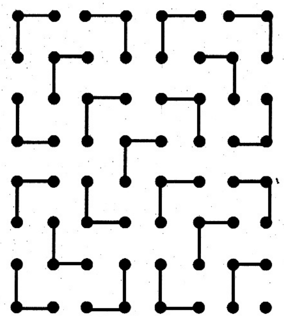


Fig 4

12. Let A and B be finite subsets of integers. Define

$A+B=\{a+b : a \in A,\ b \in B\}$

Prove that $|A+B| \geq |A|+|B|-1$ 10

Solution :

Let $A = \{a_1 < a_2 < a_3 < .... < a_r\}$

$B = \{b_1 < b_2 < b_3 < .... < b_s\}$

Thus the following chain contains r + s –1 different elements of the set A + B.

$a_1+b_1 < a_1+b_2 < ....< a_1+b_s < a_2+b_s < .... < a_r+b_s$

This proves the bounds of the problem.

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